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A Functional Analysis Method of Estimating the Settling Time of Time-Varying Systems*

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SUMMARY

In this paper a functional analysis formulation is used to determine approximately the settling time of a class of time-varying feed-back systems. A computational algorithm to determine the settling time is also presented.

1. Introduction

In the case of a feedback control system, particularly a regulator, it is often important that the system, when perturbed from an initial equilibrium state returns to it or to within a specified neighbourhood of it in the shortest possible time. In practice, the exact determination of the "settling time" is very difficult especially if the system is time-varying and/or nonlinear. In such a case it may be worthwhile to obtain an *estimate* of the settling time. Apart from its use in analyzing a specific system, a knowledge of the approximate settling times of a number of systems will enable the designer to choose the "best" system. As is well known, the designer is frequently required to select the "best" system from among a list of suitable candidates.

To the best knowledge of the authors, no published results on this problem seem to be available. One investigation, which appears to be somewhat related is the work of Letov [1] concerning the problem of "control quality". Letov's discussion is mostly qualitative and is restricted to time-invariant systems. He does not give any computational procedure.

In this paper the estimation of the settling time of a class of linear time-varying feedback control systems is considered. A precise mathematical formulation for a single input-single output linear system is discussed first. The generalization of the formulation to multi-variable systems is straightforward. An approximate solution using functional analysis as well as a simple computational algorithm are presented. The application of the algorithm is illustrated by means of an example. The extension of the method to nonlinear systems is indicated.

2. Statement and Formulation of the Problem

Consider the feedback control system shown in Fig. 1. z is the reference input. y is the output. x is the "error". K_1 and K_2 represent linear time-varying systems. They can be considered to be the kernels of linear integral operators. λ is a real number and represents the gain of the feedback loop.

The relationship between the input and the error of the system is described, in general, by the following integral equation

$$x(t) = \int_{-\infty}^{t} k_1(t,\tau)\,z(\tau)\,d\tau + \lambda \int_{-\infty}^{t} k_2(t,\tau)\,x(\tau)\,d\tau\,. \tag{2.1}$$

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Equation 2.1 is a linear Volterra integral equation of the second kind.

The settling time problem can be stated as follows.

The system shown in Fig. 1 is perturbed from the initial equilibrium state (say $x=0$) at $t=0$ by a perturbation z and the perturbation is withdrawn at a subsequent time $t=t_0>0$. Given an arbitrary $\varepsilon>0$ and that $\max|z(t)|_{t\in[0,t_0]}\leqq\delta_1$ and also that $z(t)\equiv 0$ for t outside the interval $[0, t_0]$, what is the minimum time required for the absolute value of the error signal to reach the ε-neighbourhood of the equilibrium state and stay inside the neighbourhood thereafter?

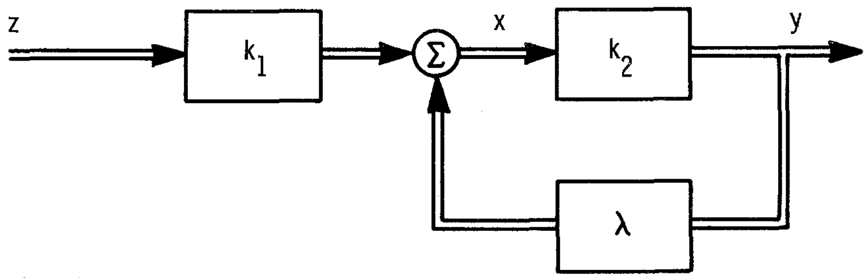

Figure 1.

In this discussion, it is assumed that the system is asymptotically stable. A set theoretic formulation of this problem can be stated as follows.

If we define a set A as

$$A(t_0,\varepsilon)=\{r\in[t_0,\infty]:|x(t)|\leqq\varepsilon \text{ for all } t\geqq r\} \tag{2.2}$$

then the determination of the settling time reduces to the determination of the infimum of $A(t_0,\varepsilon)$. Since we have assumed that the system is asymptotically stable, the infimum of $A(t_0,\varepsilon)$ always exists for an arbitrary $\varepsilon>0$ as is shown in the next section.

3. Existence of inf $A(t_0, \varepsilon)$

Taking the absolute value on both sides of equation 2.1, we obtain

$$|x(t)|\leqq\int_0^t|k_1(t,\tau)|\,|z(\tau)|\,d\tau+|\lambda|\int_0^t|k_2(t,\tau)|\,|x(\tau)|\,d\tau \qquad t\geqq 0 \tag{3.1}$$

Let $z(t)\equiv 0$ for all $t>t_0$ and let

$$\max_{t\in[0,t_0]}|z(t)|=\delta_1$$

and

$$\max_{t\in[0,t_0]}|x(t)|=\delta_2(\delta_1).$$

For $t>t_0$, we have

$$|x(t)|\leqq\delta_1\int_0^{t_0}|k_1(t,\tau)|\,d\tau+|\lambda|\,\delta_2\int_0^{t_0}|k_2(t,\tau)|\,d\tau+|\lambda|\int_{t_0}^{t}|k_2(t,\tau)|\,|x(\tau)|\,d\tau\,. \tag{3.2}$$

We assume that there exists an $\alpha_1>0$ and two bounded, continuous non-negative functions $h_1(t)$ and $h_2(t)$ such that

$$|k_1(t,\tau)|\leqq e^{-\alpha_1(t-\tau)}h_1(\tau)$$

and (3.3)

$$|k_2(t,\tau)|\leqq e^{-\alpha_1(t-\tau)}h_2(\tau)$$

for all τ and t satisfying $0\leqq\tau\leqq t\leqq\infty$.

Then, for all $t>t_0$, inequality 3.2 reduces to

$$\phi(t) \leqq \beta_1 + \beta_2 \int_{t_0}^{t} h_2(\tau)\phi(\tau)d\tau \tag{3.4}$$

where $\phi(t) = |x(t)|\, e^{\alpha_1 t}$

$$\beta_1 = \int_{t_0}^{t} e^{\alpha_1 \tau}[\delta_1 h_1(\tau) + |\lambda|\,\delta_2 h_2(\tau)]d\tau$$

and

$$\beta_2 = |\lambda|$$

Applying Gronwall's lemma [2] to inequality 3.4, we obtain

$$|x(t)| \leqq \beta_1 \exp\left\{\beta_2 \int_{t_0}^{t} h_2(\tau)\mathrm{d}\tau - \alpha_1 t\right\} \text{ for all } t \geqq t_0 \,. \tag{3.5}$$

It is clear from the inequality 3.5 that $\lim_{t\to\infty} |x(t)| \to 0$, if $\overline{\lim}\, h_2(t) \to c$, where $0 \leqq c < \alpha_1/\beta_2$. If $h_2(t)$ satisfies this condition then the set $A(t_0, \varepsilon)$ defined in 2.2 is non-empty and its infimum is finite. In other words, there exists a finite time T^*, such that $|x(t)| \leqq \varepsilon$ for all $t \geqq T^*$. A constructive proof of the foregoing statement is given in the appendix.

4. Determination of the Settling Time

Although the existence of inf $A(t_0, \varepsilon)$ has been proved its determination is not easy. Consequently in order to determine the settling time let us define another set $B(t_0, \varepsilon)$ as follows:

$$B(t_0, \varepsilon) = \{r \in [t_0, \infty] : x_1(t) \leqq \varepsilon \text{ for all } t \geqq r\} \tag{4.1}$$

where

$$x_1(t) = \beta_1 \exp\left\{\beta_2 \int_{t_0}^{t} h_2(\tau)d\tau - \alpha_1 t\right\}. \tag{4.2}$$

It is clear that $B(t_0, \varepsilon) \subset A(t_0, \varepsilon)$ and inf $B(t_0, \varepsilon) \geqq$ inf $A(t_0, \varepsilon)$. Inf $B(t_0, \varepsilon)$ can be easily determined and in most cases it can be considered to be a fairly good estimate of the settling time of the system.

The set $B(t_0, \varepsilon)$ can be expressed in another form:

$$B(t_0, \varepsilon) = \left\{r \in [t_0, \infty) : t \geqq a_1 \int_{t_0}^{t} h_2(\tau)d\tau + a_2 \text{ for all } t \geqq r\right\} \tag{4.3}$$

where $a_1 = \beta_2/\alpha_1$ and $a_2 = (1/\alpha_1)\ln(\beta_1/\varepsilon)$.

Determination of inf $B(t_0, \varepsilon)$. *Case* 1. If the inequality

$$t \geqq a_1 \int_{t_0}^{t} h_2(\tau)d\tau + a_2 \tag{4.4}$$

is satisfied for all $t \geqq t_0$, then inf $B(t_0, \varepsilon) = t_0$.

Case 2. If inequality 4.4 is not satisfied, then

$$\inf B(t_0, \varepsilon) = \max_{t \in [t_0, \infty]} (S_1 \backslash S_2)$$

where

$$S_1 = \left\{t \in [t_0, \infty] : t = a_1 \int_{t_0}^{t} h_2(\tau)d\tau + a_2\right\} \tag{4.5}$$

and $S_2 \subset S_1$ is defined as

$$S_2 = \left\{t \in S_1 : t - \eta \geqq a_1 \int_{t_0}^{t-\eta} h_2(\tau)d\tau + a_2 \text{ for sufficiently small } \eta > 0\right\} \tag{4.6}$$

The set S_2 defined above may be empty or non-empty.

The set S_2 is non-empty whenever the equation

$$t - a_1 \int_{t_0}^{t} h_2(\tau) d\tau - a_2 = 0 \tag{4.7}$$

has a continuum of roots and/or has roots of even multiplicity. For example, if both t_i and t_f are in S_1 and $t_i < t_f$ then a necessary condition that the interval $[t_i, t_f] \subset S_2$ is that $h_2(t) = 1/a_1$ for all $t \in [t_i, t_f]$. However, since $h_2(t)$ is related to the kernel of a physical system such a situation will not be very common.

If S_2 is empty, then the infimum of set B is given by the *largest* value of t which satisfies equation 4.7. If however, S_2 is non-empty, then a small quantity $\pm\eta$ can be added to a solution T^* of the equation 4.7 to determine whether or not T^* is an element of S_2. If T^* is not an element of S_2, then it may be the desired solution; if T^* is an element of S_2 then a sufficiently large positive quantity η can be subtracted from T^* to continue the iteration process which is required to solve equation 4.7. This iterative or successive approximation process is described in the course of the proof of a proposition given in the appendix. It is important to observe that whether S_2 is empty or non-empty, an estimate of the settling time of the system is given by one of the appropriate solutions of equation 4.7.

5. An Example

As an illustration of the above technique, let us consider an example of a separable time-varying system (Fig. 2). $r(t)$ is a *known* multiplicative disturbance at the input terminals of a time-

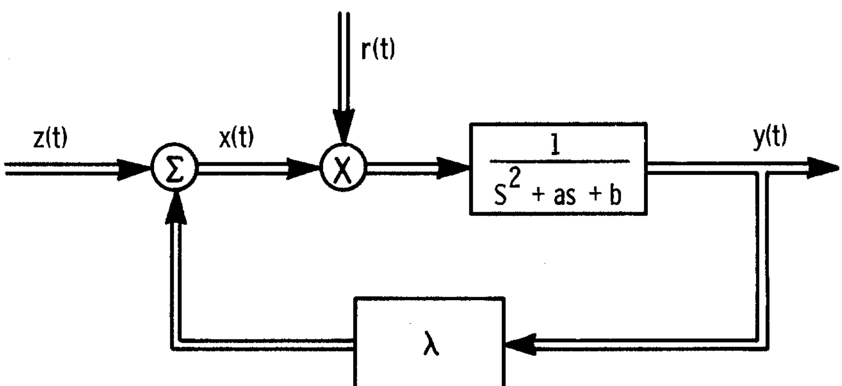

Figure 2.

invariant second order differential system. Comparing Figs. 1 and 2 we have the following relations.

$$k_1(t, \tau) = \delta(t-\tau)$$

$$k_2(t, \tau) = \frac{e^{-\alpha_1(t,\tau)} \sin \alpha_2(t-\tau)}{\alpha_2} r(t) \tag{5.2}$$

where $\alpha_1 = \frac{1}{2}a$

$$a_2 = \sqrt{b - \frac{a^2}{4}}, \quad \text{assuming} \quad b - \frac{a^2}{4} > 0\,. \tag{5.3}$$

Let

$$r(t) = \left(\frac{\mu_0 + \mu_1 t^2}{1+t^2} + \mu_2 \cos \omega t\right). \tag{5.4}$$

The numerical computations were carried out on an IBM 360/67 computer using Fortran IV language. The following numerical values were used: $a=2$; $b=8$; $\mu_0=1$; $\mu_1=1$; $\mu_2=1.5$; $\lambda=-1$ and $\omega=0.61$ radians per second; $\varepsilon=10^{-2}$.

The estimated value T^* of the settling time was found to be 6.5 seconds as against its exact value of 4.5 seconds. The computations were repeated with $\lambda = +1$ and $\omega = 3.66$ rps and other parameters unchanged. In this case the estimated value was 6.7 seconds and the exact value was 3.7 seconds.

The computational algorithm for T^* seems to yield reasonably good (although pessimistic) estimates of the settling time. It is however realized that the closeness or disparity between the estimated and exact values may depend upon the relative sizes of the parameters used. But it seems reasonable to expect the algorithm to give a fairly accurate estimate in most cases.

6. Multivariable Systems

The technique described in this paper applies to multivariable systems also. If $z(t)$ is an m-vector and $x(t)$ is an n-vector, the kernels k_1 and k_2 are $n \times m$ and $n \times n$ matrices respectively. The elements of k_1 and k_2 are functions of t and τ over the triangle $\triangle = \{t, \tau - \infty \leqq \tau \leqq t \leqq \infty\}$. λ is a scalar or a diagonal matrix with elements $(\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n)$. The symbol $|.|$ may be interpreted as any combination of the standard norms defined in Euclidean n-space. For example $|x(t)|$ may be taken as either

$$\text{(i)} \quad |x(t)| = \max_i |x_i(t)| \quad \text{or} \quad \text{(ii)} \ |x(t)| = \sqrt{\sum_{i=1}^{n} |x_i(t)|^2} \tag{6.1}$$

If the elements of k (k_1 or k_2) are represented by $k_{ij}(t, \tau)$, then $|k(t, \tau)|$ may be chosen to be either

$$\text{(i)}' \quad |k(t, \tau)| = \max_{ij} |k_{ij}(t, \tau)| \tag{6.2}$$

or

$$\text{(ii)}' \quad |k(t, \tau)| = \sqrt{\sum_i |k_{ij}(t, \tau)|^2} \tag{6.3}$$

for all $t, \tau \in \Delta$.

The first set of norms (i) and (i)′ may yield a sharper estimate of the settling time T^*.

7. Nonlinear Systems

The method presented in this paper can be applied to certain nonlinear systems as well. For instance, we can include a zero-memory nonlinear element in the forward loop to precede the linear system represented by K_2 in Fig. 1. In other words the settling time can be estimated for systems represented by the integral equation

$$x(t) = z(t) + \lambda \int_{-\infty}^{t} k_2(t, \tau) g(\tau, x(\tau)) d\tau \tag{7.1}$$

for all g satisfying the conditions

$$g(t, 0) = 0 \ \text{for all} \ t \tag{7.2}$$

and

$$|g(t, u)| \leqq a(t)|u| \ \text{for all} \ u \in R. \tag{7.3}$$

The function $a(t) \geqq 0$ is required to be continuous and bounded for all $t \in R$ and $K_2(t, \tau)\, a(\tau)$ must satisfy a property similar to 3.3 satisfied by $K_2(t, \tau)$ alone in the linear case.

Appendix

Proposition: Let

$$t = a_1 \int_{t_0}^{t} h_2(\tau) d\tau + a_2 . \tag{A.1}$$

For every pair of finite real numbers $a_1 > 0$ and $a_2 \geqq 0$ and for every continuous non-negative function $h_2(\tau)$, if $\overline{\lim}\, h_2(t) \to c$, for $0 \leqq c < 1/a_1$, then inf $B(t_0, \varepsilon)$ defined by 4.3 exists and is finite. This infimum can be determined from the solution of equation A.1 which is the same as equation 4.7 by a method of successive approximation.

Proof: Let us define a sequence $\{T_n\}$ by

$$T_{n+1} = a_1 \int_{t_0}^{T_n} h_2(\tau)d\tau + a_2 \qquad (n = 0, 1, 2, \ldots) \tag{A.2}$$

For an arbitrary choice of T_0, the sequence T_n may converge to any of the intermediate solutions of equation A.1. To determine inf B, we may choose T_0 as follows: since $\overline{\lim}\, h_2(t) \to c < 1/a_1$, there exists for every $c' \in (c, 1/a_1)$, a $T^0(c') < \infty$ such that $h(t) < c'$ for all $t \geqq T^0(c')$. Let us choose a value for T_0 such that $T^0(c') \leqq T_0 < \infty$. It is clear that $h(t) < c'$ for all $t \geqq \min(T_0, \inf B)$. Therefore

$$|T_{n+1} - T_n| \leqq \alpha |T_n - T_{n-1}| \quad \text{for} \quad n \geqq 0$$

where $\alpha = a_1 c' < 1$. For every integer $p \geqq 1$, we have

$$|T_{n+p} - T_n| \leqq \sum_{k=n}^{n+p-1} |T_{k+1} - T_k| \leqq |T_1 - T_0| \sum_{k=n}^{n+p-1} \alpha^k \leqq \frac{|T_1 - T_0|}{1-\alpha} \alpha^n \text{ for all } n \geqq 0 \,.$$

Thus $\lim_{n\to\infty} |T_{n+p} - T_n| \to 0$ for every $p \geqq 1$ and therefore $\{T_n\}$ is a Cauchy sequence in $[t_0, \infty]$. Hence T_n converges to a limit point $T^* \in [t_0, \infty)$.

That T^* is a solution of equation A.1 is proved as follows:

Let us define $g(T)$ by

$$g(T) = a_1 \int_{t_0}^{T} h_2(\tau)d\tau + a_2 . \tag{A.3}$$

Then

$$|T^* - g(T^*)| = |T^* - g(T_n) + g(T_n) - g(T^*)|$$

$$\leqq |T^* - T_{n+1}| + a_1 \left| \int_{T^*}^{T_n} h_2(\tau)d\tau \right| .$$

Since the left hand side is independent of n the inequality is true for any n. Since $T_n \to T^*$ and $h_2(t)$ is continuous and bounded we have $T^* = g(T^*)$. If $T^* \in S_2$ then the process of iteration is continued. Let us therefore consider the other possibility, namely $T^* \in S_1 \backslash S_2$. ($S_1$ and S_2 are defined earlier in section 4.)

Since every Cauchy Sequence is bounded, T^* is bounded and also

$$T^* \leqq \frac{a_2' - \alpha T_0}{1-\alpha} < \infty$$

where $a_2' = a_2 + a_1 \int_{t_0}^{T_0} h_2(\tau)d\tau < \infty$.

That T^* is actually the desired inf B is proved as follows: Suppose T^* is not the inf B, then there exists a $t^* > T^*$ such that

$$t^* < a_1 \int_{t_0}^{t^*} h_2(\tau)d\tau + a_2 .$$

If $\{T_n\}$ is a nondecreasing sequence then $h_2(t) < c'$ for all $t > T^*$. Hence $t^* < T^* + \alpha(t^* - T^*)$ implying that $t^* < T^*$. This leads to a contradiction. If $\{T_n\}$ is a nonincreasing sequence then $t^* > T^*$ implies also that $t^* > T_0$ because

$$t > a_1 \int_{t_0}^{t} h_2(\tau)d\tau + a_2 \text{ for all } t \in [T^*, T_0] .$$

Hence from the inequality

$$t^* < a_1 \int_{t_0}^{t^*} h_2(\tau)d\tau + a_2$$

it follows that

$$t^* < T_1 + \alpha(t^* - T_0) \leqq T_0 + \alpha(t^* - T_0)$$

which implies that $t^* < T_0$, thus leading to a contradiction. Therefore T^* is actually the desired inf B.

8. Acknowledgements

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